Non-commutative 4-spheres based on all Podleś 2-spheres and beyond

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Abstract

A wide class of noncommutative spaces, including 4-spheres based on all the quantum 2-spheres and suspensions of matrix quantum groups is described. For each such space a noncommutative vector bundle is constructed. This generalises and clarifies various recent constructions of noncommutative 4-spheres.

1. Recently there has been an upsurge of activity in constructing noncommutative 4-spheres and corresponding projective modules or (instanton) noncommutative vector bundles. This has been initiated by a paper by Connes and Landi [9] where, among others, a family of isospectral noncommutative 4-spheres labelled by a deformation parameter of unit modulus has been constructed. Another family of q-deformed noncommutative 4-spheres labelled by a real deformation parameter q was defined via a suspension of the quantum group $SU_q(2)$ in [12]. These have been followed up in [19], where a two-parameter family of noncommutative 4-spheres that include the Connes-Landi one as a special case, was introduced as a noncommutative two dimensional suspension of one of Podleś 2-spheres [18]. In a different direction, motivated by the classical Hopf fibering $S^7 \to S^4$, a family of noncommutative 4-spheres was defined in [2] as coinvariants of the coaction of $SU_q(2)$ (viewed as a coisotropic subgroup of $U_q(4)$) on the quantum 7-sphere, i.e., as a quantum quotient space in the sense of [3]. In all the above cases, projective modules were constructed with a projector given in terms of a 4×4-matrix with entries from the noncommutative space, and the corresponding Chern-Connes character was computed.

The aim of the present note is twofold. First we would like to generalise the construction in [19] by describing a class of noncommutative 4-spheres obtained by a two-dimensional noncommutative suspension of all the Podleś 2-spheres. Second we would like to propose a general construction of noncommutative manifolds based on the standard formulation of the matrix quantum group $GL_q(n)$ (the FRT-construction). The q-deformed 4-sphere in [12] can then be viewed as a special case of this construction with $n = 2$.

2. Algebra of functions on the quantum sphere $A(S^2_{q,s})$ [18] is defined as a polynomial algebra generated by $1, x, y, z$ subject to the relations

$$zx = q^2 xz, \quad yz = q^2 zy, \quad xy = (z-1)(z+s^2), \quad yx = (q^2 z - 1)(q^2 z + s^2), \quad (1)$$

where q, s are complex parameters, $q \neq 0$, $s^2 \neq -1$. A commutative algebra of functions on the 2-sphere can be identified with $A(S^2_{q,s})$ with $q = 1$ and $s = 0$. From a purely algebraic point of view there is no restriction on the values of deformation parameters s and q. In all cases $A(S^2_{q,s})$ is a subalgebra of the algebra of functions on the quantum

group $SL_q(2)$. If q^2 and s^2 are real then $A(S^2_{q,s})$ is a $*$-algebra with involution given by $z^* = z$, $x^* = -y$. A C^*-algebra $C(S^2_{q,s})$ corresponding to $A(S^2_{q,s})$ is defined if q is real, nonzero and $-1 < q < 1$, and either $s \in [0,1]$ or else $s^2 = -q^{2n}$, $n \in \mathbf{N}$. In the case $s^2 = -q^{2n}$ the quantum sphere is a finite dimensional C^*-algebra isomorphic to the full matrix algebra $\mathrm{Mat}_n(\mathbf{C})$, which can be interpreted as a deformation of the fuzzy sphere [16]. Thus it is termed a q-*fuzzy sphere* and denoted by $S^2_{q,n}$. $S^2_{q,n}$ is no longer a subalgebra of $SU_q(2)$. On an algebraic level (with q a root of unity), $S^2_{q,n}$ has been shown in [1] to describe D-branes of open string in the $SU(2)$ Wess-Zumino-Witten model, and was recently studied in [14].

The algebraic noncommutative 4-sphere $A(S^4_{p,q,s})$, corresponding to the algebra $A(S^2_{q,s})$ is a noncommutative algebra generated by $1, \xi, \eta, \zeta, U, V$ subject to the relations

$$\zeta\xi = q^2\xi\zeta, \quad \eta\zeta = q^2\zeta\eta, \quad \xi U = pU\xi, \quad V\xi = p\xi V,$$

$$\eta V = pV\eta, \quad U\eta = p\eta U, \quad UV = VU, \quad U\zeta = \zeta U, \quad V\zeta = \zeta V, \tag{2}$$

$$\xi\eta = (\zeta - 1)(\zeta + s^2) + UV, \quad \eta\xi = (q^2\zeta - 1)(q^2\zeta + s^2) + UV,$$

where $p, q \neq 0$, $s^2 \neq -1$ are complex parameters. The algebra $A(S^4_{p,q,s})$ can be made into a $*$-algebra with $\zeta^* = \zeta$, $\xi^* = -\eta$ and $U^* = V$, provided q^2, s^2 are real and p is a pure phase, i.e., $p = \exp(2\pi i\theta)$, where $\theta \in [0,1)$. The corresponding C^*-algebra is denoted by $C(S^4_{q,\theta,s})$. Explicitly, $C(S^4_{q,\theta,s})$ is defined by the operator (supremum) norm closure over all admissible $*$-representations of a polynomial involutive algebra which has a presentation with generators ξ, ζ, U and relations

$$\zeta\xi = q^2\xi\zeta, \quad \xi U = pU\xi, \quad U^*\xi = p\xi U^*, \quad UU^* = U^*U, \quad U\zeta = \zeta U,$$

$$\xi\xi^* + (\zeta - 1)(\zeta + s^2) + UU^* = 0, \quad \xi^*\xi + (q^2\zeta - 1)(q^2\zeta + s^2) + UU^* = 0, \tag{3}$$

with $-1 < q < 1$, $0 < s \leq 1$ and $p = \exp(2\pi i\theta)$. The quantum 4-sphere $S^4_{q,\theta}$ in [19] is isomorphic to $C(S^4_{q^2,\theta,1})$. Unitary representations $\pi_{c,\pm}$ of $C(S^4_{q,\theta,s})$ in a Hilbert space with basis $|k,l\rangle$ with $k \in \mathbf{N}$, $l \in \mathbf{Z}$ are labelled by a complex number c, $|c| \leq s$, and are given by

$$\pi_{c,\pm}(U)|k,l\rangle = c|k, l+1\rangle, \qquad \pi_{c,\pm}(U^*)|k,l\rangle = c^*|k, l-1\rangle,$$

$$\pi_{c,\pm}(\zeta)|k,l\rangle = \alpha_\pm q^{2(k-1)}|k,l\rangle,$$

$$\pi_{c,\pm}(\xi)|k,l\rangle = p^l\omega_{\pm,k}|k+1,l\rangle, \qquad \pi_{c,\pm}(\xi^*)|k+1,l\rangle = p^{-l}\omega_{\pm,k}|k,l\rangle,$$

where

$$\alpha_\pm = \frac{1}{2}\left(1 - s^2 \pm \sqrt{(s^2+1)^2 - 4|c|^2}\right), \quad \omega_{\pm,k} = \sqrt{(1 - \alpha_\pm q^{2k})(s^2 + \alpha_\pm q^{2k}) - |c|^2},$$

$k = 0, 1, 2, \ldots$. Note that if $s = 1$ these representations involve representations of the quantum sphere $C(S^2_{q,1})$ on the Hilbert space spanned by $|k\rangle$, $k \in \mathbf{N}$, thus reducing to representations of the type discussed in [19].

In the case $s^2 = -q^{2n}$, the C^*-algebra $C(S^4_{q,\theta,s})$ reduces to $C(S^2_{q,s})$ and thus is iso-morphic to the full matrix algebra $\mathrm{Mat}_n(\mathbf{C})$. Note, however, that on a purely algebraic level it does make sense to define $A(S^4_{q,\theta,s})$ corresponding to the q-fuzzy sphere $S^2_{q,n}$. A study of such a noncommutative space might be of interest to string theory.

One can easily construct an example of a projector associated to the noncommutative algebra of functions on the quantum 4-sphere $A(S^4_{p,q,s})$. This defines a noncommutative vector bundle (projective module) over $A(S^4_{p,q,s})$. Consider the following 4×4 matrix with entries from $A(S^4_{p,q,s})$,

$$e = \frac{1}{1+s^2}\begin{pmatrix} 1-\zeta & 0 & U & \xi \\ 0 & 1-q^2\zeta & -\eta & -pV \\ V & \xi & s^2+\zeta & 0 \\ -\eta & -p^{-1}U & 0 & s^2+q^2\zeta \end{pmatrix}.$$

One easily checks that $e^2 = e$, hence it is a projector as claimed. Furthermore in the case when $A(S^4_{q,\theta,s})$ is a $*$-algebra and thus including $C(S^4_{q,\theta,s})$, this projector is self-adjoint, i.e., $e^* = e$ ($*$ in $\mathrm{Mat}_4(A(S^4_{q,\theta,s}))$ combines $*$ in $A(S^4_{q,\theta,s})$ with matrix transposition). Thus e defines a projective module $E = \{\mathbf{v}e \mid \mathbf{v} = (v_1, v_2, v_3, v_4), \ v_i \in A(S^4_{p,q,s})\}$ or a noncommutative vector bundle over $A(S^4_{q,\theta,s})$ [7]. Such vector bundles are classified by the Chern-Connes classes in cyclic homology. The components of the Chern-Connes class of E are defined by

$$ch_n(E) = c_n \sum_{i_1\ldots i_{2n+1}} (e - \frac{1}{2})_{i_1 i_2} \otimes \bar{e}_{i_2 i_3} \otimes \bar{e}_{i_3 i_4} \otimes \cdots \otimes \bar{e}_{i_{2n+1} i_1},$$

where $\bar{e}$ is e projected down to the nonunital part of the algebra of functions on the quantum sphere $A(S^4_{p,q,s})/\mathbf{C}1$, and c_n are normalisation factors. The Chern-Connes

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class is a cocycle in the cyclic homology of $A(S^4_{p,q,s})$. Clearly $ch_0(E) = 0$ and

$$ch_1(E) \;\propto\; \frac{1}{(1+s^2)^3}(q^2-1)(\zeta\otimes(U\otimes V - V\otimes U) + U\otimes(V\otimes\zeta - \zeta\otimes V)$$
$$+ V\otimes(\zeta\otimes U - U\otimes\zeta)).$$

Note that up to normalisation and slightly different conventions $ch_1(E)$ has the same form as the ones computed in [12] and [19]. In particular, within the range of s the first factor can be absorbed in the normalisation, and thus $ch_1(E)$ does not essentially depend on s. Furthermore, $ch_1(E) = 0$ if and only if $q = \pm 1$.

Following the same method as in [19] one can define a projective module $\tilde{E}$ over $A(S^4_{p,q,s})$ for which $ch_0(\tilde{E}) = ch_1(\tilde{E}) = 0$, provided one formally adjoins a (self-adjoint) central element $Z = \sqrt{UV}$ to $A(S^4_{p,q,s})$. In the case of the corresponding C^*-algebra this can be done by considering a suitable infinite series. To construct noncommutative vector bundle $\tilde{E}$ one uses the charge 1 magnetic monopole projector for all Podleś two-spheres constructed in [5] [6], and defines a projector $\tilde{e}$ in $\mathrm{Mat}_4(A(S^4_{p,q,s}))$ by

$$\tilde{e} = \frac{1}{2(1+s^2)}\begin{pmatrix} 1+s^2+2Z & 0 & 1-s^2-2\zeta & 2\xi \\ 0 & 1+s^2+2Z & -2\eta & s^2-1+2q^2\zeta \\ 1-s^2-2\zeta & 2\xi & 1+s^2-2Z & 0 \\ -2\eta & s^2-1+2q^2\zeta & 0 & 1+s^2-2Z \end{pmatrix}. \qquad (4)$$

One can easily find directly that $\tilde{e}$ is a projector and that $ch_0(\tilde{E}) = ch_1(\tilde{E}) = 0$. There is no need to do it here, for a general justification of this fact is provided below. Clearly $\tilde{e}$ is self-adjoint with respect to the $*$-structure on $A(S^4_{q,\theta,s})$, whenever defined. Note also that since $\tilde{e}$ depends on Z, ζ, ξ and η (and does not depend on U and V separately) it might be viewed as defined on the quantum 3-sphere [11].

3. The projector $\tilde{e}$ is a special case of more general construction which provides one with a wide range of noncommutative algebras and corresponding projectors. Consider an algebra A generated by 1, an $n \times n$ matrix of generators $\mathbf{t} = (t_{ij})$ and by an additional generator Z. Consider another $n \times n$ matrix $\tilde{\mathbf{t}}$ of elements of A. In case A is a $*$-algebra one requires $Z^* = Z$ and $\tilde{\mathbf{t}} = \mathbf{t}^*$. Then a $2n \times 2n$ matrix e with entries from A given in the block form by

$$e = \frac{1}{2}\begin{pmatrix} 1+Z & \mathbf{t} \\ \tilde{\mathbf{t}} & 1-Z \end{pmatrix}, \qquad (5)$$

is a (self-adjoint) projector provided Z is central in A and

$$\tilde{\mathbf{t}}\mathbf{t} = \mathbf{t}\tilde{\mathbf{t}} = 1 - Z^2. \tag{6}$$

The first two components of the Chern-Connes character of the corresponding noncommutative vector bundle E over A come out as $ch_0(E) = 0$ and

$$\begin{aligned} ch_1(E) \ &\propto \ \sum_{ij} (t_{ij} \otimes (\tilde{t}_{ji} \otimes Z - Z \otimes \tilde{t}_{ji}) + \tilde{t}_{ji} \otimes (Z \otimes t_{ij} - t_{ij} \otimes Z) \\ &+ Z \otimes (t_{ij} \otimes \tilde{t}_{ji} - \tilde{t}_{ji} \otimes t_{ij})). \end{aligned} \tag{7}$$

This is precisely the method used to obtain $\tilde{e}$ above. As proven in [5] [6] the following matrix with entries from $A(S^2_{q,s})$,

$$f = \frac{1}{1+s^2} \begin{pmatrix} 1-z & x \\ -y & s^2 + q^2 z \end{pmatrix},$$

is a projector in $\mathrm{Mat}_2(A(S^2_{q,s}))$, i.e., $f^2 = f$. The matrix f describes a noncommutative line bundle associated to the Dirac monopole principal bundle (Hopf fibration) over $A(S^2_{q,s})$. The connection defined by f (the Grassmann connection) is a gauge field of the q-deformed Dirac monopole. The fact that $f^2 = f$ implies that

$$\mathbf{t} = \tilde{\mathbf{t}} = \frac{2}{1+s^2} \begin{pmatrix} 1-\zeta & \xi \\ -\eta & s^2 + q^2\zeta \end{pmatrix} - 1$$

satisfies $\mathbf{t}^2 = 1 - (2Z/(1+s^2))^2$, where $Z^2 = UV$ in (suitably extended) $A(S^4_{p,q,s})$, i.e., the condition (6) holds. Matrix $\tilde{e}$ in (4) has precisely the block form (5) (with Z replaced by $2Z/(1+s^2)$). Since $\mathbf{t} = \tilde{\mathbf{t}}$ the first component of the corresponding Chern-Connes character vanishes by (7). In fact again using that $\mathbf{t} = \tilde{\mathbf{t}}$ one easily finds that $ch_k(\tilde{E}) = 0$, k$= 0, 1, 2, \ldots$

A rich source of algebras with projectors of the block form (5) and non-trivial Chern-Connes characters is provided by the standard, FRT-construction of matrix quantum groups. Recall that the FRT-construction [13] associates an algebra $A(R)$ to any invertible $n^2 \times n^2$ solution R of the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. The algebra $A(R)$ is generated by 1 and an $n \times n$ matrix $\mathbf{t}$ subject to the following RTT-relations

$$R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R, \tag{8}$$

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where $\mathbf{t}_1 = \mathbf{t} \otimes 1$, $\mathbf{t}_2 = 1 \otimes \mathbf{t}$. In each of such algebras there is a central element known as the *quantum determinant*, $\det_q(\mathbf{t})$ ($\det_q(\mathbf{t})$ is grouplike if $A(R)$ is equipped with the matrix bialgebra structure). Quantum determinant can be computed explicitly in the framework of braided groups (cf. [17]). Furthermore, by using quantum minors in $\mathbf{t}$ one can construct a matrix $\tilde{\mathbf{t}}$ with the property $\mathbf{t}\tilde{\mathbf{t}} = \tilde{\mathbf{t}}\mathbf{t} = \det_q(\mathbf{t})$. Now one can define an algebra A by adjoining a central element Z to $A(R)$ which is required to satisfy $\det_q(\mathbf{t}) = 1 - Z^2$. The resulting algebra A has an associated noncommutative vector bundle over itself with a projector given by (5).

As an example consider an algebra associated to the standard solution of the quantum Yang-Baxter equation corresponding to $GL_q(n)$,

$$R = q^{-1/n}\left(q\sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1})\sum_{i > j} E_{ij} \otimes E_{ji} \right),$$

where E_{ij} are the usual matrix units. The elements of matrix $\tilde{\mathbf{t}}$ come out as

$$\tilde{t}_{ij} = (-q)^{i-j}\sum_{\sigma \in S_{n-1}} (-q)^{\ell(\sigma)} t_{j_1 i_{\sigma(1)}} \cdots t_{j_{n-1} i_{\sigma(n-1)}},$$

where S_{n-1} is the permutation group and $\{i_1, \ldots, i_{n-1}\} = \{1, \ldots, i-1, i+1, \ldots n\}$, and $\{j_1, \ldots, j_{n-1}\} = \{1, \ldots, j-1, j+i, \ldots n\}$. In the case q-real one can define a consistent $*$-structure by imposing $t_{ij}^* = \tilde{t}_{ji}$. The resulting algebra A is a $*$-algebra given by the RTT-relations (8) and $t_{ij}^* t_{jk} = t_{ij} t_{jk}^* = (1 - Z^2)\delta_{ik}$, and has associated self-adjoint projector e in $\mathrm{Mat}_{2n}(A)$ of the form in equation (5). In the case $n = 2$ this is exactly the quantum 4-sphere S_q^4 and e the corresponding projector introduced in [12].

4. In this note we have extended some results of recent papers [12] and [19]. We introduced a wide class of examples of noncommutative spaces with implicit quantum group symmetry. It is hoped that these examples will help in deciding how axioms for a noncommutative manifold in [8] might be modified in order to include examples based on quantum groups. On the other hand it would be interesting, and we belive indeed desired, to study whether (some of) the introduced vector bundles can be viewed as bundles associated to quantum (coalgebra) principal bundles [4] in the way analogous to the q-deformed Dirac monopole in [6] or a bundle over the quantum 4-sphere in [2],

and whether the constructed projectors correspond to strong connections [15][10] on such principal bundles.

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